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LETTER TO THE EDITOR

Supersymmetric quantum mechanics of fermions minimally coupled to gauge fields

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Abstract. The supersymmetric extension of the system consisting of a relativistic scalar particle interacting with a gauge field is discussed. The construction is based on the ground-state wavefunction representation for supersymmetric quantum mechanics. We exploit the fact that any system invariant under spacetime reparametrisations has a vanishing Hamiltonian.

In a recent article, Ui [1] has shown that a Dirac particle coupled to a gauge field in three spacetime dimensions possesses a supersymmetry analogous to Witten's supersymmetric quantum mechanics (SSQM) [2]. In this letter we show that Ui's conclusion can be generalised to any number of dimensions, provided the supersymmetry generators are appropriately defined. The extension to include non-Abelian gauge fields is discussed as well.

The construction given below is based on a scheme originally proposed by Gozzi [3] to discuss the hidden supersymmetry of one-dimensional quantum mechanical systems. This scheme was later extended to an arbitrary number of degrees of freedom [4, 5]. Here we will briefly summarise the method which is discussed in greater detail in [5], and then apply it to the case of a relativistic particle in the background of a gauge field.

Because of the invariance under time reparametrisation, the classical Hamiltonian of a generally covariant system can be written as a linear combination of first-class constraints $\mathcal{H}_a \approx 0$ in the form [6]:

$$H = \int \sum_a N^a(x)\mathcal{H}_a(x)\, \mathrm{d}^{D-1}x \approx 0 \tag{1}$$

where $a = \perp; 1, \ldots, n \leqslant D-1$; D is the number of spacetime dimensions; N^a are Lagrange multipliers. The constraint $\mathcal{H}_\perp$ generates timelike reparametrisation and is responsible for the dynamical evolution. The remaining constraints $\mathcal{H}_i$ generate spacelike reparametrisations that leave the states of the system unchanged; these are generators of gauge transformations with no dynamical effect [6]. Hamiltonians of the form (1) describe systems like the relativistic particle interacting with various fields, strings, membranes and gravitation.

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Dirac's prescription [7] to construct the corresponding quantum theory calls for the substitution of the first-class constraint functions $\mathscr{H}_a$ by operators $\hat{\mathscr{H}}_a$, and the constraint equations $\mathscr{H}_a \approx 0$ by conditions on the physical states, ψ, of the Hilbert space

$$\hat{H}_a\psi = 0. \tag{2}$$

This implies, in particular

$$\hat{H}\psi = 0 \tag{3}$$

and, therefore, the physical states of a quantum system of this type can be viewed as belonging to the (degenerate) 'ground state' of a system in $D+1$ dimensions (the new extra dimension being a fictitious time, λ, in the Schrödinger equation $\hat{H}\psi = \mathrm{i}(\partial/\partial\lambda)\psi$). This interpretation, however, is not quite correct as it stands: the operator $\hat{H}$ for any of the systems mentioned above is generically hyperbolic and not elliptic, as it should be for a genuine Schrödinger equation. This fact, due to the Minkowskian signature of spacetime, is responsible for the non-uniqueness of ψ and makes the spectrum of $\hat{H}$ unbounded from below and above. However, passing to Euclidean space via a Wick rotation $t \to \mathrm{i}t = t_\mathrm{E}$, turns $\hat{H}$ into an elliptic differential operator and the analogy with quantum mechanics can be exploited to construct the supersymmetric extension of the bosonic theory following the pattern of [3-5]. Finally, the supersymmetric extension of the original theory in Minkowski space is obtained by undoing the Wick rotation at the end, $t_\mathrm{E} \to t_\mathrm{E}/\mathrm{i} = t$.

Now let us consider the case of a relativistic particle interacting with an Abelian gauge field in D-dimensional Minkowski space. The classical Hamiltonian constraint is, in Euclidean space,

$$\mathscr{H}_\mathrm{E} = \tfrac{1}{2}[(p_\mu - eA_\mu)^2 + U^2] \approx 0 \tag{4}$$

where $U(x)$ is a real scalar function, which is included here for greater generality ($U(x) = m = \text{constant}$, for the simplest case). According to Dirac's prescription, we write, for the quantum theory,

$$\hat{\mathscr{H}}_\mathrm{E}\Psi_0 = 0. \tag{5}$$

Here $\hat{\mathscr{H}}_\mathrm{E}$ is the operator version of (4), and (5) is the Schrödinger equation for the ground state (with ground-state energy renormalised to zero). In the Schrödinger representation ($\hat{p}_\mu = \mathrm{i}\partial_\mu$), the wavefunction Ψ_0 takes the form

$$\Psi_0(x) = \exp(-V) \int \mathscr{D}C \exp\left(-\mathrm{i}e \int_C^x A_\mu(z)\, \mathrm{d}z_\mu\right) \tag{6}$$

where V is a real function related to U by the Ricatti equation:

$$\partial_\mu V \partial_\mu V - \partial_\mu\partial_\mu V = U^2. \tag{7}$$

The contour of integration C in (6) is any trajectory that reaches the point x and is chosen so as to avoid the sources of A_μ. (A different choice of contour might change Ψ_0 by a non-trivial phase (Aharonov-Bohm effect). Differentiability requires that Ψ_0 be defined on source-free open neighbourhoods only and, therefore, such contours always exist.) As usual, the gauge symmetry is guaranteed by the invariance of (5) under the simultaneous changes $A_\mu \to A_\mu + \partial_\mu\Lambda$, $\Psi_0 \to \exp(-\mathrm{i}e\Lambda)\Psi_0$, for arbitrary local functions $\Lambda(x)$. Substitution of (7) in the expression for $\hat{\mathscr{H}}_\mathrm{E}$ provides the ground-state representation

$$\hat{\mathscr{H}}_\mathrm{E} = \tfrac{1}{2}Q_\mu^\dagger Q_\mu \tag{8}$$

with

$$Q_\mu = \nabla_\mu + \partial_\mu V \tag{9a}$$

$$Q_\mu^\dagger = -\nabla_\mu + \partial_\mu V \tag{9b}$$

and

$$\nabla_\mu = \partial_\mu + \mathrm{i}eA_\mu = -\mathrm{i}(p_\mu - eA_\mu). \tag{10}$$

Next, following Gozzi [3, 4], we define the superchanges as

$$\mathcal{Q} = Q_\mu \gamma_\mu \otimes \tau_- \qquad \mathcal{Q}^\dagger = Q_\mu^\dagger \gamma_\mu \otimes \tau_+ \tag{11}$$

where $\tau_+ = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$, $\tau_- = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$ and the γ_μ are the Dirac matrices appropriate for D-dimensional Euclidean space†. This ansatz generalises the form of the supercharges proposed elsewhere for supersymmetric non-relativistic quantum mechanics (SUSYQM) [4, 9]. Following those ideas, the supersymmetric extension of (8) is

$$\hat{\mathcal{H}}_{\mathrm{E}}^{(S)} = \tfrac{1}{2}\{\mathcal{Q}^\dagger, \mathcal{Q}\}. \tag{12}$$

This new (supersymmetric) Hamiltonian can be directly checked to be invariant under transformations generated by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$:

$$[\mathcal{Q}, \hat{\mathcal{H}}_{\mathrm{E}}^{(S)}] = 0 = [\mathcal{Q}^\dagger, \hat{\mathcal{H}}_{\mathrm{E}}^{(S)}] \tag{13}$$

and, obviously

$$[\hat{\mathcal{H}}_{\mathrm{E}}^{(S)}, \hat{\mathcal{H}}_{\mathrm{E}}^{(S)}] = 0. \tag{14}$$

Relations (12)-(14) are the graded algebras of the supersymmetric system consisting of a (relativistic) spin-$\frac{1}{2}$ particle interacting with an external electromagnetic field. From (10)-(12) the explicit form of the superHamiltonian constraint can be found. After rotating back to Minkowski space, it becomes

$$\hat{\mathcal{H}}^{(S)} = \frac{1}{2}\begin{bmatrix} Q_\mu^\dagger Q^\mu - \sigma^{\mu\nu}L_{\mu\nu} - \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu} & 0 \\ 0 & Q^\mu Q_\mu^\dagger + \sigma^{\mu\nu}L_{\mu\nu} - \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu} \end{bmatrix} \tag{15}$$

where $L_{\mu\nu} =: \partial_\mu V p_\nu - \partial_\nu V p_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the absence of an electromagnetic field ($F_{\mu\nu} = 0$), equation (15) reduces to the Hamiltonian constraint for a spin-$\frac{1}{2}$ particle in a scalar potential $U(x)$ ('variable mass') [5]. The term $\pm\sigma^{\mu\nu}L_\mu$ generalises the spin-orbit coupling found in non-relativistic SUSYQM [4]. The interaction term $\frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}$ is the usual Pauli coupling which exhibits the correct magnetic moment for an electron. (It is curious that supersymmetry naturally requires the gyromagnetic factor, g, to be exactly 2. The fact that radiative corrections make g slightly different from this value can be viewed as an indication that this supersymmetry is dynamically broken when the electromagnetic field is also quantised.)

We now summarise the changes that occur in the above discussion if the gauge field is non-Abelian. The vector potential A_μ is to be replaced by the matrix-valued field $\mathbb{A}_\mu = A_\mu^a \mathbb{T}_a$, where $\mathbb{T}_a$ are the generators of some compact Lie group (e.g., SU(N)). Then, replacing the covariant derivative ∇_μ by $\boldsymbol{\nabla}_\mu = \partial_\mu + \mathrm{i}g[\mathbb{A}_\mu, \ ]$, the form of $\mathcal{H}^{(S)}$ appropriate for this case is found as‡

$$\hat{\mathcal{H}}^{(S)} = \tfrac{1}{2}\,\mathrm{Tr}\{\tilde{\mathcal{Q}}, \mathcal{Q}\} \tag{16}$$

† The Euclidean Dirac matrices are Hermitian and satisfy $\{\gamma_\mu^{(\mathrm{E})}, \gamma_\nu^{(\mathrm{E})}\} = 2\delta_{\mu\nu}$, $\mu, \nu = 0, 1, \ldots, D-1$. Their commutator is $[\gamma_\mu^{(\mathrm{E})}, \gamma_\nu^{(\mathrm{E})}] = 2\mathrm{i}\sigma_{\mu\nu}^{(\mathrm{E})}$, where $\sigma_{\mu\nu}^{(\mathrm{E})}$ is also Hermitian. For conventions see, e.g., [8].

‡ Note that the Dirac matrices in Minkowski space $\gamma_\mu = (\mathrm{i}\gamma_0^{(\mathrm{E})}, \gamma_i^{(\mathrm{E})})$ are not Hermitian, so that the supercharges $\mathcal{Q}_{(\mathrm{E})}$ and $\mathcal{Q}_{(\mathrm{E})}^\dagger$ are mapped into $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ respectively, where $\tilde{\mathcal{Q}} \neq \mathcal{Q}^\dagger$ in Minkowski space.

where the trace is performed over gauge group indices ($\mathrm{Tr}(\mathbb{T}_a \mathbb{T}_b) = \delta_{ab}$). The rest of the algebra closes as before and the explicit form of (16) is that of (15) where ∇_μ is replaced by $\boldsymbol{\nabla}_\mu$ and $F_{\mu\nu}$ by $\mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + \mathrm{i}g[\mathbb{A}_\mu, \mathbb{A}_\nu]$. In other words, supersymmetry and gauge symmetry do not mix.

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References

[1] Ui H 1984 *Prog. Theor. Phys.* **72** 192
[2] Witten E 1981 *The Unity of the Fundamental Interactions* ed A Zichichi (New York: Plenum)
[3] Gozzi E 1983 *Phys. Lett.* **129B** 432
[4] Gamboa J and Zanelli J 1985 *Phys. Lett.* **165B** 91
[5] Gamboa J and Zanelli J 1986 *Proc. 5th Chilean Symp. of Physics (Sociedad Chilena de Física, Santiago, 1986)*; 1986 *Proc. Latin American School of Physics, Mexico 1986*; 1987 *Preprint, CECS-UCH* Quantum mechanics of supersymmetric strings
[6] Hanson A J, Regge T and Teitelboim C 1976 *Constrained Hamiltonian Systems* (Rome: Accademia dei Licei)
[7] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Academic)
[8] Nepomechie R 1984 *Ann. Phys., NY* **158** 67
[9] Ui H 1984 *Prog. Theor. Phys.* **72** 813